

Some applications of eta-quotients

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Abstract

We show that every modular form on $\Gamma_0(2^n)$ ($n \geq 2$) can be expressed as a sum of eta-quotients. Furthermore, we construct a primitive generator of the ring class field of the order of conductor $4N$ ($N \geq 1$) in an imaginary quadratic field in view of the special value of certain eta-quotient.

1 Introduction

For a positive integer N let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

This group acts on the complex upper-half plane \mathbb{H} as fractional linear transformations, and gives rise to the modular curve $X_0(N) = \Gamma_0(N) \backslash \mathbb{H}^*$, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ [10, Chapter 1]. We denote its (meromorphic) function field by $\mathbb{C}(X_0(N))$. Then it is well-known that $\mathbb{C}(X_0(N)) = \mathbb{C}(j(\tau), j(N\tau))$, where $j(\tau)$ is the elliptic modular function whose Fourier expansion with respect to $q = e^{2\pi\tau}$ has integer Fourier coefficients as follows:

$$j(\tau) = 1/q + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \quad (\tau \in \mathbb{H})$$

[8, §4.1 and §6.4, Theorem 7].

We define the *Dedekind eta-function* $\eta(\tau)$ by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (\tau \in \mathbb{H}), \quad (1)$$

and call a product of the form $\prod_{d|N} \eta(d\tau)^{m_d}$ with $m_d \in \mathbb{Z}$ an *eta-quotient*. Ono suggested the following question [9, Problem 1.68]: Classify the spaces of modular forms which are generated

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by eta-quotients. And, Choi [1] recently showed that if $X_0(N)$ is of genus zero (that is, in the cases $N = 1, \dots, 10, 12, 13, 16, 18, 25$), then every modular form on $\Gamma_0(N)$ can be expressed as a \mathbb{C} -linear combination of certain eta-quotients.

Denote by $\mathcal{R}_0(N)$ the integral closure of the polynomial ring $\mathbb{C}[j(\tau)]$ in the function field $\mathbb{C}(X_0(N))$. In this paper, we shall first construct explicit generators of the ring $\mathcal{R}_0(2^n)$ ($n \geq 2$) over \mathbb{C} in terms of eta-quotients (Theorem 3.6). As its corollary we can give an answer to Ono's question when $N = 2^n$ (Corollary 3.7).

On the other hand, let K be an imaginary quadratic field of discriminant d_K , and set

$$\tau_K = \begin{cases} (-1 + \sqrt{d_K})/2 & \text{if } d_K \equiv 1 \pmod{4}, \\ \sqrt{d_K}/2 & \text{if } d_K \equiv 0 \pmod{4}. \end{cases} \quad (2)$$

Let $H_{K,N}$ be the ring class field of the order of conductor N in K . As a consequence of the main theorem of the theory of complex multiplication, we have

$$H_{K,N} = K(j(N\tau_K)) \quad (3)$$

[8, §10.3, Theorem 5]. We shall show that if $N \equiv 0 \pmod{4}$, then $256\eta(N\tau_K)^8/\eta((N/4)\tau_K)^8$ becomes a primitive generator of $H_{K,N}$ over K (Theorem 4.5).

For a number field F , we denote by \mathcal{O}_F its ring of integers. We shall further verify the structure of $\mathcal{O}_{H_{K,2^n}}[1/2]$ ($n \geq 3$) over $\mathcal{O}_{H_{K,4}}[1/2]$ in view of the special values of eta-quotients generating $\mathcal{R}_0(2^n)$ (Theorem 4.8). To this end, we shall introduce an explicit version of Shimura's reciprocity law.

2 Modular forms and functions

We shall briefly examine the modularity of eta-quotients, Weierstrass functions and Siegel functions.

LEMMA 2.1. *Let N be a positive integer. Assume that a family of integers $\{m_d\}_{d|N}$, where d runs over all positive divisors of N , satisfies the following conditions:*

- (i) $\sum_{d|N} m_d$ is even.
- (ii) $\sum_{d|N} dm_d \equiv \sum_{d|N} (N/d)m_d \equiv 0 \pmod{24}$.
- (iii) $\prod_{d|N} d^{m_d}$ is a square in \mathbb{Q} .

Then, the eta-quotient $\prod_{d|N} \eta(d\tau)^{m_d}$ is a meromorphic modular form of weight $(1/2)\sum_{d|N} m_d$ on $\Gamma_0(N)$ having rational Fourier coefficients with respect to q .

PROOF. See [9, Theorem 1.64] and the definition (1). □

REMARK 2.2. Every eta-quotient has neither zeros nor poles on \mathbb{H} by the definition (1).

For a lattice Λ in \mathbb{C} , the *Weierstrass \wp -function* relative to Λ is defined by

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right) \quad (z \in \mathbb{C}).$$

This is a meromorphic function on z and is periodic with respect to Λ . Moreover, for $z_1, z_2 \in \mathbb{C} \setminus \Lambda$ we have the assertion

$$\wp(z_1; \Lambda) = \wp(z_2; \Lambda) \iff z_1 \equiv \pm z_2 \pmod{\Lambda} \quad (4)$$

[11, Chapter IV, §3].

Let $N \geq 2$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$. We define

$$\wp_{\mathbf{v}}(\tau) = \wp(v_1\tau + v_2; [\tau, 1]) \quad (\tau \in \mathbb{H}).$$

It is a meromorphic modular form of weight 2 on the principal congruence subgroup $\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \pm I_2 \pmod{N}\}$. Furthermore, it depends only on $\pm \mathbf{v} \pmod{\mathbb{Z}^2}$ and satisfies the following transformation formula: If $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\wp_{\mathbf{v}}(\tau) \circ \alpha = (c\tau + d)^2 \wp_{\alpha^T \mathbf{v}}(\tau), \quad (5)$$

where α^T stands for the transpose of α [8, §6.2].

LEMMA 2.3. *Let $n \geq 2$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in (1/2)\mathbb{Z}^2 \setminus \mathbb{Z}^2$.*

(i) *If $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(2^{n-1})$, then we have*

$$\wp_{\mathbf{v}}(2^{n-1}\tau) \circ \alpha = (c\tau + d)^2 \wp_{\begin{bmatrix} v_1 + (c/2^{n-1})v_2 \\ v_2 \end{bmatrix}}(2^{n-1}\tau).$$

(ii) *$\wp_{\mathbf{v}}(2^{n-1}\tau)$ is a meromorphic modular form of weight 2 on $\Gamma_0(2^n)$.*

PROOF. (i) Note that $a, d \equiv 1 \pmod{2}$. We derive that

$$\begin{aligned} \wp_{\mathbf{v}}(2^{n-1}\tau) \circ \alpha &= \wp_{\mathbf{v}}(2^{n-1}(a\tau + b)/(c\tau + d)) \\ &= \wp_{\mathbf{v}}(\tau) \circ \begin{bmatrix} 2^{n-1}a & 2^{n-1}b \\ c & d \end{bmatrix} \\ &= \wp_{\mathbf{v}}(\tau) \circ \begin{bmatrix} a & 2^{n-1}b \\ c/2^{n-1} & d \end{bmatrix} \circ \begin{bmatrix} 2^{n-1} & 0 \\ 0 & 1 \end{bmatrix} \\ &= (((c/2^{n-1})\tau + d)^2 \wp_{\begin{bmatrix} a & 2^{n-1}b \\ c/2^{n-1} & d \end{bmatrix}^T \mathbf{v}}(\tau)) \circ \begin{bmatrix} 2^{n-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{by (5)} \\ &= ((c/2^{n-1})(2^{n-1}\tau) + d)^2 \wp_{\begin{bmatrix} av_1 + (c/2^{n-1})v_2 \\ 2^{n-1}bv_1 + dv_2 \end{bmatrix}}(2^{n-1}\tau) \\ &= (c\tau + d)^2 \wp_{\begin{bmatrix} v_1 + (c/2^{n-1})v_2 \\ v_2 \end{bmatrix}}(2^{n-1}\tau) \\ &\quad \text{because } \begin{bmatrix} av_1 + (c/2^{n-1})v_2 \\ 2^{n-1}bv_1 + dv_2 \end{bmatrix} \equiv \begin{bmatrix} v_1 + (c/2^{n-1})v_2 \\ v_2 \end{bmatrix} \pmod{\mathbb{Z}^2}. \end{aligned}$$

(ii) Moreover, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(2^n)$. Since $c \equiv 0 \pmod{2^n}$, we get $(c/2^{n-1})v_2 \in \mathbb{Z}$. So we obtain by (i) that

$$\wp_{\mathbf{v}}(2^{n-1}\tau) \circ \alpha = (c\tau + d)^2 \wp_{\mathbf{v}}(2^{n-1}\tau).$$

This shows that $\wp_{\mathbf{v}}(2^{n-1}\tau)$ is a meromorphic modular form of weight 2 on $\Gamma_0(2^n)$. \square

For a positive integer N let $\mathbb{C}(X(N))$ be the function field of the modular curve $X(N) = \Gamma(N) \backslash \mathbb{H}^*$. Then, $\mathbb{C}(X(N))$ is a Galois extension of $\mathbb{C}(X(1)) = \mathbb{C}(X_0(1))$ whose Galois group is naturally isomorphic to $\bar{\Gamma}(1)/\bar{\Gamma}(N)$, where $\bar{\Gamma}(N) = \langle \Gamma(N), \pm I_2 \rangle / \{\pm I_2\}$ [10, p.31].

Let $N \geq 2$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$. We define the *Siegel function* $g_{\mathbf{v}}(\tau)$ on \mathbb{H} by

$$g_{\mathbf{v}}(\tau) = -q^{(1/2)(v_1^2 - v_1 + 1/6)} e^{\pi i v_2 (v_1 - 1)} (1 - q^{v_1} e^{2\pi i v_2}) \prod_{n=1}^{\infty} (1 - q^{n+v_1} e^{2\pi i v_2}) (1 - q^{n-v_1} e^{-2\pi i v_2}). \quad (6)$$

Then $g_{\mathbf{v}}(\tau)$ and $g_{\mathbf{v}}(\tau)^{12N}$ belong to $\mathbb{C}(X(12N^2))$ and $\mathbb{C}(X(N))$, respectively [4, Chapter 3, Theorems 5.2 and 5.3]. Furthermore, if $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \in \mathbb{Z}^2$, then $g_{\mathbf{v}}(\tau)$ satisfies the translation formula

$$g_{\mathbf{v}+\mathbf{s}}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 v_2 - s_2 v_1)} g_{\mathbf{v}}(\tau) \quad (7)$$

[4, pp.28–29].

LEMMA 2.4. *We have the following relations.*

(i) *If $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ such that $\mathbf{u} \not\equiv \pm \mathbf{v} \pmod{\mathbb{Z}^2}$, then we have*

$$\wp_{\mathbf{u}}(\tau) - \wp_{\mathbf{v}}(\tau) = -g_{\mathbf{u}+\mathbf{v}}(\tau) g_{\mathbf{u}-\mathbf{v}}(\tau) \eta(\tau)^4 / g_{\mathbf{u}}(\tau)^2 g_{\mathbf{v}}(\tau)^2.$$

(ii) $g_{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}(\tau) g_{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}(\tau) g_{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}(\tau) = 2e^{\pi i/4}$.

(iii) $g_{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}(\tau) = -\eta(\tau/2)^2 / \eta(\tau)^2$.

PROOF. (i) See [4, p.51].

(ii) See [7, Lemma 2.6(ii)].

(iii) We see that

$$\begin{aligned} g_{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}(\tau) &= -q^{-1/24} (1 - q^{1/2}) \prod_{n=1}^{\infty} (1 - q^{n+1/2}) (1 - q^{n-1/2}) \quad \text{by the definition (6)} \\ &= -q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{(1/2)(2n-1)})^2 \\ &= -q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{(1/2)n})^2 / \prod_{n=1}^{\infty} (1 - q^{(1/2)(2n)})^2 \\ &= -\eta(\tau/2)^2 / \eta(\tau)^2 \quad \text{by the definition (1)}. \end{aligned}$$

\square

3 Integral closures in modular function fields

In this section, we shall find explicit generators of $\mathcal{R}_0(2^n)$ over \mathbb{C} by using eta-quotients. As a consequence of this result, we shall further show that every modular form on $\Gamma_0(2^n)$ can be written as a sum of eta-quotients.

LEMMA 3.1. *Let N be a positive integer. The ring $\mathcal{R}_0(N)$ consists of weakly holomorphic (that is, holomorphic on \mathbb{H}) functions in $\mathbb{C}(X_0(N))$.*

PROOF. Note first that every weakly holomorphic function in $\mathbb{C}(X_0(1)) = \mathbb{C}(X(1))$ is a polynomial in $j(\tau)$ over \mathbb{C} [8, §5.2, Theorem 2].

Let $h(\tau) \in \mathcal{R}_0(N)$, so $h(\tau)$ satisfies

$$h(\tau)^m + P_{m-1}(j(\tau))h(\tau)^{m-1} + \cdots + P_0(j(\tau)) = 0$$

for some $m \geq 1$ and $P_{m-1}(X), \dots, P_0(X) \in \mathbb{C}[X]$. Dividing both sides by $h(\tau)^m$ we get

$$1 + P_{m-1}(j(\tau))(1/h(\tau)) + \cdots + P_0(j(\tau))(1/h(\tau))^m = 0. \quad (8)$$

Suppose that $h(\tau)$ has a pole at a point τ_0 in \mathbb{H} , so $1/h(\tau)$ has a zero at the point. But, if we insert $\tau = \tau_0$ into (8), then we get a contradiction $1 = 0$. Hence $h(\tau)$ must be weakly holomorphic.

Conversely, let $h(\tau) \in \mathbb{C}(X_0(N))$ be weakly holomorphic. Since $\Gamma(N) \leq \Gamma_0(N) \leq \Gamma(1)$, $\mathbb{C}(X_0(N))$ is an intermediate field of the extension $\mathbb{C}(X(N))/\mathbb{C}(X(1))$. Thus $h(\tau) \circ \gamma$ for $\gamma \in \Gamma(1)$ represent all the Galois conjugates of $h(\tau)$ over $\mathbb{C}(X(1)) = \mathbb{C}(X_0(1))$. It follows that every coefficient of $\min(h(\tau), \mathbb{C}(X_0(1)))$ is also weakly holomorphic, and hence belongs to $\mathbb{C}[j(\tau)]$. This shows that $h(\tau) \in \mathcal{R}_0(N)$, and completes the proof. \square

For each $n \geq 3$ we define

$$h_n(\tau) = \frac{\wp\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau) - \wp\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)}{\wp\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau) - \wp\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)} \quad (\tau \in \mathbb{H}). \quad (9)$$

It belongs to $\mathbb{C}(X_0(2^n))$ by Lemma 2.3(ii), and has neither zeros nor poles on \mathbb{H} by (4). Hence it is in $\mathcal{R}_0(2^n)^\times$ by Lemma 3.1.

LEMMA 3.2. *We have $h_n(\tau) = \eta(2^{n-2}\tau)^{12} / \eta(2^{n-1}\tau)^4 \eta(2^{n-3}\tau)^8$.*

PROOF. We find that

$$\begin{aligned}
h_n(\tau) &= \frac{\wp\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau) - \wp\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)}{\wp\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau) - \wp\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)} \\
&= \frac{g\left[\begin{smallmatrix} 1/2 \\ 1 \end{smallmatrix}\right](2^{n-1}\tau)g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-1}\tau)\eta(2^{n-1}\tau)^4/g\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)^2g\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)^2}{g\left[\begin{smallmatrix} 1/2 \\ 1 \end{smallmatrix}\right](2^{n-2}\tau)g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-2}\tau)\eta(2^{n-2}\tau)^4/g\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)^2g\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)^2} \quad \text{by Lemma 2.4(i)} \\
&= \frac{g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-1}\tau)^2\eta(2^{n-1}\tau)^4/g\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)^2g\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-1}\tau)^2}{g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-2}\tau)^2\eta(2^{n-2}\tau)^4/g\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)^2g\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right](2^{n-2}\tau)^2} \quad \text{by (7)} \\
&= \frac{g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-1}\tau)^4\eta(2^{n-1}\tau)^4}{g\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right](2^{n-2}\tau)^4\eta(2^{n-2}\tau)^4} \quad \text{by Lemma 2.4(ii)} \\
&= \frac{\eta(2^{n-2}\tau)^{12}}{\eta(2^{n-1}\tau)^4\eta(2^{n-3}\tau)^8} \quad \text{by Lemma 2.4(iii).}
\end{aligned}$$

□

REMARK 3.3. Due to Lemma 3.2, one can also use Lemma 2.1 to show that $h_n(\tau)$ belongs to $\mathcal{R}_0(2^n)^\times$ and has rational Fourier coefficients with respect to q .

LEMMA 3.4. *Let $n \geq 3$. We have $\mathcal{R}_0(2^n) = \mathcal{R}_0(2^{n-1})[h_n(\tau)]$.*

PROOF. Since $h_n(\tau) \in \mathcal{R}_0(2^n)^\times$, we obviously have $\mathcal{R}_0(2^{n-1})[h_n(\tau)] \subseteq \mathcal{R}_0(2^n)$.

Note that

$$\text{Gal}(\mathbb{C}(X_0(2^n))/\mathbb{C}(X_0(2^{n-1}))) \simeq \overline{\Gamma}_0(2^{n-1})/\overline{\Gamma}_0(2^n) = \{I_2, \begin{bmatrix} 1 & 0 \\ 2^{n-1} & 1 \end{bmatrix}\},$$

where $\overline{\Gamma}_0(N) = \langle \Gamma_0(N), \pm I_2 \rangle / \{\pm I_2\}$ ($N \geq 1$) [10, p.31]. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. Since $\wp_{\mathbf{v}}(2^{n-2}\tau)$ is a meromorphic modular forms of weight 2 on $\Gamma_0(2^{n-1})$ by Lemma 2.3(ii), we have

$$\wp_{\mathbf{v}}(2^{n-2}\tau) \circ \begin{bmatrix} 1 & 0 \\ 2^{n-1} & 1 \end{bmatrix} = (2^{n-1}\tau + 1)^2 \wp_{\mathbf{v}}(2^{n-2}\tau).$$

Furthermore, we get by Lemma 2.3(i) that

$$\wp_{\mathbf{v}}(2^{n-1}\tau) \circ \begin{bmatrix} 1 & 0 \\ 2^{n-1} & 1 \end{bmatrix} = (2^{n-1}\tau + 1)^2 \wp_{\begin{bmatrix} v_1+v_2 \\ v_2 \end{bmatrix}}(2^{n-1}\tau).$$

Thus we obtain

$$h_n(\tau) \circ \begin{bmatrix} 1 & 0 \\ 2^{n-1} & 1 \end{bmatrix} = -h_n(\tau), \quad (10)$$

and hence $\mathbb{C}(X_0(2^n)) = \mathbb{C}(X_0(2^{n-1}))(h_n(\tau))$. Furthermore, the fact $h_n(\tau) \in \mathcal{R}_0(2^n)^\times$ and (10) imply that

$$h_n(\tau)^2 \in \mathcal{R}_0(2^{n-1})^\times. \quad (11)$$

Now, let $h(\tau) \in \mathcal{R}_0(2^n)$. Since $\mathbb{C}(X_0(2^n))$ is a quadratic extension of $\mathbb{C}(X_0(2^{n-1}))$ generated by $h_n(\tau)$, we can express $h(\tau)$ as

$$h(\tau) = c_0(\tau) + c_1(\tau)h_n(\tau) \text{ for some } c_0(\tau), c_1(\tau) \in \mathbb{C}(X_0(2^{n-1})). \quad (12)$$

Put $h'(\tau) = h(\tau) \circ \begin{bmatrix} 1 & 0 \\ 2^{n-1} & 1 \end{bmatrix}$, which also lies in $\mathcal{R}_0(2^n)$ by Lemma 3.1. By (10) and (12) we get a system

$$\begin{bmatrix} h(\tau) \\ h'(\tau) \end{bmatrix} = \begin{bmatrix} 1 & h_n(\tau) \\ 1 & -h_n(\tau) \end{bmatrix} \begin{bmatrix} c_0(\tau) \\ c_1(\tau) \end{bmatrix}$$

and find

$$c_0(\tau) = (h(\tau) + h'(\tau))/2 \text{ and } c_1(\tau) = (h(\tau)h_n(\tau) + h'(\tau)(-h_n(\tau)))/2h_n(\tau)^2,$$

which belong to $\mathcal{R}_0(2^{n-1})$ by (10) and (11). This shows that $h(\tau) \in \mathcal{R}_0(2^{n-1})[h_n(\tau)]$, and hence $\mathcal{R}_0(2^n) \subseteq \mathcal{R}_0(2^{n-1})[h_n(\tau)]$. Therefore we achieve $\mathcal{R}_0(2^n) = \mathcal{R}_0(2^{n-1})[h_n(\tau)]$, as desired. \square

Let $\mathbb{Q}(X_0(N))$ be the subfield of $\mathbb{C}(X_0(N))$ consisting of functions with rational Fourier coefficients with respect to q . Let

$$g_{0,4}(\tau) = \eta(4\tau)^8 / \eta(\tau)^8,$$

which belongs to $\mathbb{Q}(X_0(4))$ by Lemma 2.1.

LEMMA 3.5. *We have the following structures on $\Gamma_0(4)$:*

- (i) $\mathbb{C}(X_0(4)) = \mathbb{C}(g_{0,4}(\tau))$ and $\mathbb{Q}(X_0(4)) = \mathbb{Q}(g_{0,4}(\tau))$.
- (ii) $\mathcal{R}_0(4) = \mathbb{C}[g_{0,4}(\tau), \eta(2\tau)^{24} / \eta(4\tau)^{16} \eta(\tau)^8, \eta(4\tau)^{16} \eta(\tau)^8 / \eta(2\tau)^{24}]$.

PROOF. (i) See [5, Table 2 and Lemma 4.1] and [3, Remark 3.4].

(ii) See [3, Theorem 3.3(i) and Remark 3.4]. \square

THEOREM 3.6. *Let $n \geq 3$. We have*

$$\mathcal{R}_0(2^n) = \mathbb{C}[g_{0,4}(\tau), \eta(2\tau)^{24} / \eta(4\tau)^{16} \eta(\tau)^8, \eta(4\tau)^{16} \eta(\tau)^8 / \eta(2\tau)^{24}, h_3(\tau), \dots, h_n(\tau)].$$

PROOF. This result follows from Lemmas 3.4 and 3.5(ii). \square

COROLLARY 3.7. *Every modular form on $\Gamma_0(2^n)$ ($n \geq 2$) can be expressed as a sum of eta-quotient.*

PROOF. Let $h(\tau)$ be a modular form of weight $2k$ ($k \geq 0$) on $\Gamma_0(2^n)$. Since $\eta(2\tau)^4 / \eta(4\tau)^8$ is a meromorphic modular form of weight -2 on $\Gamma_0(4)$ and is weakly holomorphic by Lemma 2.1 and Remark 2.2, the product $h(\tau)(\eta(2\tau)^4 / \eta(4\tau)^8)^k$ belongs to $\mathcal{R}_0(2^n)$. Thus it can be expressed as a sum of eta-quotients by Lemma 3.5(ii) and Theorem 3.6, and hence $h(\tau)$ itself is a sum of eta-quotients, too. \square

4 Generation of ring class fields

For a positive integer N let \mathcal{F}_N be the field of functions in $\mathbb{C}(X(N))$ whose Fourier coefficients with respect to $q^{1/N}$ lie in $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$. It is well-known that \mathcal{F}_N is a Galois extension of $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ whose Galois group is isomorphic to

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\} \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$

Let $h(\tau) = \sum_{n>-\infty} c_n q^{n/N}$ be a function in \mathcal{F}_N . The matrix $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ with $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ acts on $h(\tau)$ by

$$h(\tau) \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} = \sum_{n>-\infty} c_n^{\sigma_d} q^{n/N}, \quad (13)$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ given by $\zeta_N^{\sigma_d} = \zeta_N^d$. And, $\alpha \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $h(\tau)$ by

$$h(\tau)^\alpha = h(\tau) \circ \tilde{\alpha}, \quad (14)$$

where $\tilde{\alpha}$ is any preimage of α with respect to the natural reduction $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ [8, §6.3, Theorem 3].

Let K be an imaginary quadratic field with τ_K as in (2). We denote by $K_{N\mathcal{O}_K}$ its ray class field modulo $N\mathcal{O}_K$. Furthermore, we let H_K be its Hilbert class field and $H_{K,N}$ be the ring class field of the order of conductor N in K . As consequences of the main theorem of the theory of complex multiplication we obtain

$$\begin{aligned} K_{N\mathcal{O}_K} &= K(h(\tau_K) \mid h(\tau) \in \mathcal{F}_N \text{ is finite at } \tau_K), \\ H_{N,K} &= K(h(\tau_K) \mid h(\tau) \in \mathbb{Q}(X_0(N)) \text{ is finite at } \tau_K) \end{aligned} \quad (15)$$

([8, §10.1, Corollary to Theorem 2] and [5, Theorem 3.4]).

LEMMA 4.1 (Shimura's reciprocity law). *Let $\min(\tau_K, \mathbb{Q}) = X^2 + BX + C$ and*

$$W_{K,N} = \left\{ \gamma = \begin{bmatrix} t - Bs & -Cs \\ s & t \end{bmatrix} \mid t, s \in \mathbb{Z}/N\mathbb{Z} \text{ such that } \gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \right\}.$$

(i) *The group $W_{K,N}$ gives rise to the surjection*

$$\begin{aligned} W_{K,N} &\rightarrow \mathrm{Gal}(K_{N\mathcal{O}_K}/H_K) \\ \gamma &\mapsto (h(\tau_K) \mapsto h^\gamma(\tau_K) \mid h(\tau) \in \mathcal{F}_N \text{ is finite at } \tau_K) \end{aligned}$$

whose kernel is

$$\begin{cases} \{\pm I_2, \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \{\pm I_2, \pm \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \{\pm I_2\} & \text{otherwise.} \end{cases}$$

(ii) The subgroup $\{tI_2 \mid t \in (\mathbb{Z}/N\mathbb{Z})^\times\}$ of $W_{K,N}$ gives rise to the isomorphism

$$\begin{aligned} \{tI_2 \mid t \in (\mathbb{Z}/N\mathbb{Z})^\times\} / \{\pm I_2\} &\rightarrow \text{Gal}(K_{N\mathcal{O}_K}/H_{K,N}) \\ tI_2 &\mapsto (h(\tau_K) \mapsto h^{tI_2}(\tau_K) \mid h(\tau) \in \mathcal{F}_N \text{ is finite at } \tau_K). \end{aligned}$$

PROOF. (i) See [10, Theorem 6.31 and Proposition 6.34] and [12, §3].

(ii) See [6, Proposition 3.8]. \square

LEMMA 4.2. Let $\mathcal{O} = [N\tau_K, 1]$ be the order of conductor N in K . We have the degree formula

$$[H_{K,N} : K] = \frac{[H_K : K]N}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \prod_{p|N} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right),$$

where

$$\left(\frac{d_K}{p}\right) = \begin{cases} \text{the Legendre symbol} & \text{if } p \text{ is an odd prime,} \\ \text{the Kronecker symbol} & \text{if } p = 2. \end{cases}$$

PROOF. See [2, Theorem 7.24]. \square

LEMMA 4.3. Let $N \equiv 0 \pmod{4}$. There exist polynomials $A(X), B(X) \in \mathbb{Q}[X]$ satisfying

$$(i) \quad j(N\tau) = A(g_{0,4}((N/4)\tau)) / B(g_{0,4}((N/4)\tau)),$$

$$(ii) \quad B(g_{0,4}((N/4)\tau_K)) \neq 0.$$

PROOF. Since $j(4\tau) \in \mathbb{Q}(X_0(4))$ and $\mathbb{Q}(X_0(4)) = \mathbb{Q}(g_{0,4}(\tau))$ by Lemma 3.5(i), we can express $j(4\tau)$ as $j(4\tau) = A(g_{0,4}(\tau)) / B(g_{0,4}(\tau))$ for some relatively prime polynomials $A(X), B(X) \in \mathbb{Q}[X]$. It follows that $j(N\tau) = A(g_{0,4}((N/4)\tau)) / B(g_{0,4}((N/4)\tau))$.

On the other hand, one can readily check by Lemma 2.1 and Remark 2.2 that the function

$$g_{0,4}((N/4)\tau) = \eta(N\tau)^8 / \eta((N/4)\tau)^8$$

belongs to $\mathbb{Q}(X_0(N))$ and is weakly holomorphic. Thus we obtain by (15) that

$$g_{0,4}((N/4)\tau_K) \in H_{K,N}. \tag{16}$$

Suppose that $B(g_{0,4}((N/4)\tau_K)) = 0$. Since $j(N\tau)$ is weakly holomorphic, we must have $A(g_{0,4}((N/4)\tau_K)) = 0$. But this implies that $\min(g_{0,4}((N/4)\tau_K), \mathbb{Q})$ divides both $A(X)$ and $B(X)$, which yields a contradiction. Therefore $B(g_{0,4}((N/4)\tau_K))$ is nonzero. \square

LEMMA 4.4. If M is a positive integer and $\tau_0 \in \mathbb{H}$ is an imaginary quadratic argument, then the special value $M\eta(M\tau_0)^2 / \eta(\tau_0)^2$ is an algebraic integer dividing M .

PROOF. See [8, §12.2, Theorem 4]. \square

THEOREM 4.5. Let $N \equiv 0 \pmod{4}$. Then the special value $256\eta(N\tau_K)^8 / \eta((N/4)\tau_K)^8$ generates $H_{K,N}$ over K as a real algebraic integer.

PROOF. Let $A(X)$ and $B(X)$ be polynomials in $\mathbb{Q}[X]$ satisfying (i) and (ii) in Lemma 4.3. We deduce that

$$\begin{aligned}
H_{K,N} &= K(j(N\tau_K)) \quad \text{by (3)} \\
&= K(A(g_{0,4}((N/4)\tau_K))/B(g_{0,4}((N/4)\tau_K))) \quad \text{by Lemma 4.3} \\
&\subseteq K(g_{0,4}((N/4)\tau_K)) \quad \text{because } A(X), B(X) \in \mathbb{Q}[X] \\
&\subseteq H_{K,N} \quad \text{by (16)}.
\end{aligned}$$

Therefore we achieve $H_{N,K} = K(g_{0,4}((N/4)\tau_K)) = K(\eta(N\tau_K)^8/\eta((N/4)\tau_K)^8)$. Moreover, $256\eta(N\tau_K)^8/\eta((N/4)\tau_K)^8$ is an algebraic integer by Lemma 4.4 with $M = 4$ and $\tau_0 = (N/4)\tau_K$. \square

REMARK 4.6. Since $256\eta(N\tau_K)^8/\eta((N/4)\tau_K)^8$ is a real algebraic integer, its minimal polynomial over K has integer coefficients [5, Remark 4.9].

EXAMPLE 4.7. Consider the case when $K = \mathbb{Q}(\sqrt{-7})$ and $N = 12$. Let $h(\tau) = 256\eta(12\tau)^8/\eta(3\tau)^8$. Then the special value $h(\tau_K)$ generates $H_{K,N}$ over K as a real algebraic integer by Theorem 4.5.

We have $H_K = K$ [2, Theorem 12.34] and find by Lemma 4.1 that

$$\begin{aligned}
\text{Gal}(H_{K,N}/H_K) &\simeq \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} -9 & 4 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} \right\}.
\end{aligned}$$

Now, due to Lemma 4.1, (13) and (14) one can compute (by using Maple ver.15)

$$\begin{aligned}
\min(h(\tau_K), K) &= (X - h \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(\tau_K))(X - h \circ \begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix}(\tau_K))(X - h \circ \begin{bmatrix} -3 & 4 \\ -4 & 5 \end{bmatrix}(\tau_K)) \\
&\quad (X - h \circ \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}(\tau_K))(X - h \circ \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}(\tau_K))(X - h \circ \begin{bmatrix} -9 & 4 \\ 2 & -1 \end{bmatrix}(\tau_K)) \\
&\quad (X - h \circ \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}(\tau_K))(X - h \circ \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}(\tau_K)) \\
&= X^8 + 64X^7 + 2365X^6 + 5617X^5 + 1025614X^4 + 13744576X^3 \\
&\quad + 99275140X^2 + 263731264X + 1.
\end{aligned}$$

THEOREM 4.8. Let $n \geq 3$. We have

$$\mathcal{O}_{H_{K,2^n}}[1/2] = \mathcal{O}_{H_{K,4}}[1/2, h_3(\tau_K), \dots, h_n(\tau_K)].$$

PROOF. For each $m \geq 3$ we see by Lemma 3.2

$$h_m(\tau) = (\eta(2^{m-2}\tau)/\eta(2^{m-1}\tau))^4 (\eta(2^{m-2}\tau)/\eta(2^{m-3}\tau))^8.$$

Since $h_m(\tau) \in \mathcal{R}_0(2^m)^\times \cap \mathbb{Q}(X_0(2^m))$ by Remark 3.3, we obtain by (15) and Lemma 4.4 that

$$h_m(\tau_K) \in (\mathcal{O}_{H_{K,2^m}}[1/2])^\times. \quad (17)$$

So we deduce the inclusion

$$\mathcal{O}_{H_{K,2^n}} \supseteq \mathcal{O}_{H_{K,4}}[1/2, h_3(\tau_K), \dots, h_n(\tau_K)].$$

On the other hand, we see by Lemma 4.2 that $[H_{K,2^m} : H_{K,2^{m-1}}] = 2$, and find by Lemma 4.1 that

$$\text{Gal}(H_{K,2^m}/H_{K,2^{m-1}}) \simeq \{I_2, \begin{bmatrix} 1 - B2^{m-1} & -C2^{m-1} \\ 2^{m-1} & 1 \end{bmatrix}\},$$

where $\min(\tau_K, \mathbb{Q}) = X^2 + BX + C$. Decompose $\begin{bmatrix} 1 - B2^{m-1} & -C2^{m-1} \\ 2^{m-1} & 1 \end{bmatrix} \in \text{GL}_2(\mathbb{Z}/2^m\mathbb{Z})$ into

$$\begin{bmatrix} 1 - B2^{m-1} & -C2^{m-1} \\ 2^{m-1} & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 - B2^{m-1} \end{bmatrix} \begin{bmatrix} 1 - B2^{m-1} + a2^m & -C2^{m-1} + b2^m \\ 2^{m-1} + c2^m & 1 - B2^{m-1} + d2^m \end{bmatrix} \pmod{2^m}$$

for some $a, b, c, d \in \mathbb{Z}$ in order that $\begin{bmatrix} 1 - B2^{m-1} + a2^m & -C2^{m-1} + b2^m \\ 2^{m-1} + c2^m & 1 - B2^{m-1} + d2^m \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$. We derive that

$$\begin{aligned} & h_m(\tau_K) \begin{bmatrix} 1 - B2^{m-1} & -C2^{m-1} \\ 2^{m-1} & 1 \end{bmatrix} \\ = & h_m(\tau) \begin{bmatrix} 1 & 0 \\ 0 & 1 - B2^{m-1} \end{bmatrix} \begin{bmatrix} 1 - B2^{m-1} + a2^m & -C2^{m-1} + b2^m \\ 2^{m-1} + c2^m & 1 - B2^{m-1} + d2^m \end{bmatrix} (\tau_K) \quad \text{by Lemma 4.1(i)} \\ = & h_m(\tau) \begin{bmatrix} 1 - B2^{m-1} + a2^m & -C2^{m-1} + b2^m \\ 2^{m-1} + c2^m & 1 - B2^{m-1} + d2^m \end{bmatrix} (\tau_K) \quad \text{by the fact } h_m(\tau) \in \mathbb{Q}(X_0(2^m)) \text{ and (13)} \\ = & \frac{\wp \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (2^{m-1}\tau) - \wp \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (2^{m-1}\tau)}{\wp \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (2^{m-2}\tau) - \wp \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (2^{m-2}\tau)} \circ \begin{bmatrix} 1 - B2^{m-1} + a2^m & -C2^{m-1} + b2^m \\ 2^{m-1} + c2^m & 1 - B2^{m-1} + d2^m \end{bmatrix} (\tau_K) \\ & \text{by the definition (9) and (14)} \\ = & \frac{\wp \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (2^{m-1}\tau_K) - \wp \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (2^{m-1}\tau_K)}{\wp \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (2^{m-2}\tau_K) - \wp \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (2^{m-2}\tau_K)} \quad \text{by Lemma 2.3} \\ = & -h_m(\tau_K). \end{aligned}$$

This shows that $H_{K,2^m} = H_{K,2^{m-1}}(h_m(\tau_K))$, and $-h_m(\tau_K)$ is another Galois conjugate of $h_m(\tau_K)$ over $H_{K,2^{m-1}}$. Thus we get by (17) that

$$h_m(\tau_K)^2 \in (\mathcal{O}_{H_{K,2^{m-1}}}[1/2])^\times. \quad (18)$$

Now, let x be an element of $\mathcal{O}_{H_{K,2^m}}[1/2]$ and $x' \in \mathcal{O}_{H_{K,2^m}}[1/2]$ be its Galois conjugate (not necessarily distinct). Express x as $x = c_0 + c_1 h_m(\tau_K)$ for some $c_0, c_1 \in H_{K,2^{m-1}}$. Since $x' = c_0 + c_1(-h_m(\tau_K))$, we get a system

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 1 & h_m(\tau_K) \\ 1 & -h_m(\tau_K) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

It follows that

$$c_0 = (x + x')/2 \text{ and } c_1 = (xh_m(\tau_K) + x'(-h_m(\tau_K)))/2h_m(\tau_K)^2,$$

which belong to $\mathcal{O}_{H_{K,2^{m-1}}}[1/2]$ by (18). This shows that $x \in \mathcal{O}_{H_{K,2^{m-1}}}[1/2, h_m(\tau_K)]$, and hence $\mathcal{O}_{H_{K,2^m}}[1/2] \subseteq \mathcal{O}_{H_{K,2^{m-1}}}[1/2, h_m(\tau_K)]$. Hence we achieve

$$\mathcal{O}_{H_{K,2^n}}[1/2] \subseteq \mathcal{O}_{H_{K,4}}[1/2, h_3(\tau_K), \dots, h_n(\tau_K)].$$

This completes the proof. □

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